

# Spin-Josephson effect in antiferromagnetic tunnel junctions

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We study Josephson-like phenomena in a tunnel junction between two itinerant antiferromagnets. We find at the mean-field level an equilibrium current of the staggered magnetic moment through the junction that is proportional to the normal state conductance and to  $\mathbf{S}_L \times \mathbf{S}_R$  where  $\mathbf{S}_L$  and  $\mathbf{S}_R$  are the staggered magnetic moments on either sides. Microscopically, this effect comes from the coherent tunneling of spin-one charge-zero particle-hole pairs with a net wave vector equal to the antiferromagnetic one. We explain similarities and differences with the standard DC and AC Josephson effects.

One of the most striking manifestations of superconductivity is the Josephson effect, which consists in the coherent tunneling of Cooper pairs across a junction when both sides are superconductors. Interestingly, this occurs even though the Hamiltonian contains only single-electron tunneling. The resulting current is proportional to  $\sin(\varphi)$ , where  $\varphi$  is the difference between the phases of the two superconducting order parameters. In the presence of a constant potential difference, gauge invariance requires the phase difference to increase linearly with time, leading to an alternating current whose frequency depends only on universal physical constants and not on any material parameter.

Superconductivity is just one example of coherence and spontaneous symmetry breaking. The order parameter is a complex number so the broken symmetry is  $U(1)$ . The superconducting state selects a phase and the Josephson effect arises from the tendency to make the phase uniform across the tunnel junction.

Since there are many other types of order and corresponding broken symmetries, the question of the analog of the Josephson effect in such cases arises naturally [1]. Indeed, one should expect differences in order parameters across a junction to lead to a coherent tunneling of the condensed objects that exist in the broken symmetry state. This possibility is especially relevant in the context where junctions between magnetic materials are of utmost importance for spintronics. In fact, theoretical predictions have been made recently concerning the possible existence of a Josephson-like equilibrium spin current in ferromagnetic (FM) tunnel junctions in analogy with superconducting junctions [2, 3, 4]. FM long range order is a realization of spontaneous  $SO(3)$  symmetry breaking and an equilibrium spin current would result from the exchange coupling between the magnetic moments in the two leads, which favors alignment of order parameters. From a Ginzburg-Landau point of view, there is, in the functional, a term proportional to  $\mathbf{M}_L \cdot \mathbf{M}_R$  where  $\mathbf{M}_L$  and  $\mathbf{M}_R$  are, respectively, the magnetic moments on the left and on the right of the junction. The Heisenberg equations of motion thus lead to  $\frac{d\mathbf{M}_L}{dt} \sim \mathbf{M}_L \times \mathbf{M}_R$ , corresponding to a spin current.

Josephson-like phenomena between antiferromagnets

(AF) are interesting for several reasons. In the context of spintronics, it has been shown experimentally that the absence of a net angular momentum in AF results in orders of magnitude faster spin dynamics than in FM, which could expand the now limited set of applications for AF materials [5]. More generally, in AF the Néel order parameter breaks both lattice translation symmetry and  $SO(3)$  spin rotation symmetry so the situation is less straightforward than in FM from a mathematical point of view. In addition, AF are often close to superconducting phases, as is found in heavy fermions, high-temperature superconductors and layered organic superconductors. Understanding Josephson-like phenomena in AF is the first step towards more general studies with coexisting antiferromagnetic and superconducting order parameters. Generalized Josephson effects may help identify homogeneous coexistence in real materials. Studies along these lines have recently appeared for ferromagnetism coexisting with other types of order [6].

In this paper, we consider a microscopic model for single electron tunneling between itinerant AF. This microscopic calculation leads to an explicit expression for the analog of the critical current and its temperature dependence. In addition, we show that (a) Cooper-pair tunneling is replaced by tunneling of a spin-one neutral particle-hole pair (b) time dependence introduced by external magnetic fields resemble the AC Josephson effect but there are many differences because of the non-abelian nature of the problem and because spins do not couple to the gauge field but directly to the magnetic field. Finally, we briefly discuss how such effects can in principle be observed experimentally.

*The model* The Hamiltonian for a tunneling junction consisting of two leads of an AF material and an insulating barrier between them reads

$$H = H_L(c_{\mathbf{k}\sigma}^\dagger, c_{\mathbf{k}\sigma}) + H_R(d_{\mathbf{q}\sigma}^\dagger, d_{\mathbf{q}\sigma}) + H_T, \quad (1)$$

where  $H_{L(R)}$  is the Hamiltonian of the left (right) AF,  $H_T$  is the tunneling part connecting the two leads, and  $c_{\mathbf{k}\sigma}^\dagger(c_{\mathbf{k}\sigma})$  and  $d_{\mathbf{q}\sigma}^\dagger(d_{\mathbf{q}\sigma})$  are the fermion creation (annihilation) operators of the left and right leads, respectively. In the following discussion the quantum numbers  $\mathbf{k}$  and  $\mathbf{q}$  will also denote implicitly the left and right lead.

We model the AF on each side of the junction by a one-band Hubbard Hamiltonian treated in the Hartree-Fock approximation for a static spin-density wave (SDW) with wave-vector  $\mathbf{Q}$ . Without loss of generality, we assume the SDW mean field to be polarized along the spin quantization axis. Following Ref. [8], we write

$$\hat{H}_L = \sum_{\mathbf{k}\alpha} \epsilon_k c_{\mathbf{k}\alpha}^\dagger c_{\mathbf{k}\alpha} - \frac{US}{2} \sum_{\mathbf{k}\alpha\beta} c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger \sigma_{\alpha\beta}^3 c_{\mathbf{k}\beta}, \quad (2)$$

where  $\epsilon_k$  is the band dispersion,  $U$  the interaction strength,  $\sigma^3$  the third Pauli matrix, while the order parameter  $S$  is defined by  $(1/N)\langle \sum_{\mathbf{k}\alpha\beta} c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger \sigma_{\alpha\beta}^3 c_{\mathbf{k}\beta} \rangle$  with  $N$  the number of sites. This one-body Hamiltonian can be diagonalized by the Bogoliubov transformation

$$\begin{aligned} \gamma_{\mathbf{k}\alpha}^c &= u_{\mathbf{k}} c_{\mathbf{k}\alpha} + v_{\mathbf{k}} \sum_{\beta} (\sigma^3)_{\alpha\beta} c_{\mathbf{k}+\mathbf{Q}\beta}, \\ \gamma_{\mathbf{k}\alpha}^v &= v_{\mathbf{k}} c_{\mathbf{k}\alpha} - u_{\mathbf{k}} \sum_{\beta} (\sigma^3)_{\alpha\beta} c_{\mathbf{k}+\mathbf{Q}\beta}. \end{aligned} \quad (3)$$

To avoid double counting,  $\mathbf{k}$  is restricted to the magnetic zone. The superscripts  $c$  and  $v$  refer to the conduction and the valence bands split by the exchange Bragg scattering from the SDW. For simplicity, we assume perfect nesting  $\epsilon_{\mathbf{k}} = -\epsilon_{\mathbf{k}+\mathbf{Q}}$ . In this case, the coefficients of the transformation are given by  $u_{\mathbf{k}}^2 = [\frac{1}{2}(1 + \epsilon_{\mathbf{k}}/E_{\mathbf{k}})]$ ,  $v_{\mathbf{k}}^2 = [\frac{1}{2}(1 - \epsilon_{\mathbf{k}}/E_{\mathbf{k}})]$ ,  $E_{\mathbf{k}}^2 = (\epsilon_{\mathbf{k}}^2 + \Delta^2)$ , where  $\Delta = -US/2$  is the SDW gap parameter [9]. The diagonalized Hamiltonian is given by  $H = \sum_{\mathbf{k}\alpha} E_{\mathbf{k}} (\gamma_{\mathbf{k}\alpha}^{c\dagger} \gamma_{\mathbf{k}\alpha}^c - \gamma_{\mathbf{k}\alpha}^{v\dagger} \gamma_{\mathbf{k}\alpha}^v)$  where  $\sum_{\mathbf{k}}$  means that the sum extends over the magnetic zone. The single-particle energy spectrum is given by  $\pm E_{\mathbf{k}}$  and the SDW ground state for a half-filled band is defined by  $\gamma_{\mathbf{k}\alpha}^{c\dagger} |\Omega\rangle = \gamma_{\mathbf{k}\alpha}^c |\Omega\rangle = 0$ , which corresponds to:

$$|\Omega\rangle = \Pi_{\mathbf{k}\alpha}^* (v_{\mathbf{k}} c_{\mathbf{k}\alpha}^\dagger - u_{\mathbf{k}} \sum_{\beta} c_{\mathbf{k}+\mathbf{Q}\beta}^\dagger \sigma_{\beta\alpha}^3) |0\rangle \quad (4)$$

$$= \Pi_{\mathbf{k}\alpha}^* (v_{\mathbf{k}} - u_{\mathbf{k}} \sum_{\beta} c_{\mathbf{k}+\mathbf{Q}\beta}^\dagger \sigma_{\beta\alpha}^3 c_{\mathbf{k}\alpha}) c_{\mathbf{k}\alpha}^\dagger |0\rangle. \quad (5)$$

The last form makes the analogies with the BCS ground state clear. For example, there exists Andreev-like reflections at AF-N interfaces [10]. The ground state contains coherent particle-hole pairs. To clarify this, we perform a particle-hole transformation for states that are in the first magnetic-Brillouin zone. Recall that destroying an electron in a state, creates a hole in the corresponding time-reversed state. For a spinor this can be achieved by  $c_{\mathbf{k}\uparrow} \rightarrow h_{-\mathbf{k}\downarrow}^\dagger$  and  $c_{\mathbf{k}\downarrow} \rightarrow -h_{-\mathbf{k}\uparrow}^\dagger$ . The ground state then takes the form

$$|\Omega\rangle = \Pi_{\mathbf{k}}^* (v_{\mathbf{k}} - u_{\mathbf{k}} c_{\mathbf{k}+\mathbf{Q}\uparrow}^\dagger h_{-\mathbf{k}\downarrow}^\dagger) (v_{\mathbf{k}} - u_{\mathbf{k}} c_{\mathbf{k}+\mathbf{Q}\downarrow}^\dagger h_{-\mathbf{k}\uparrow}^\dagger) |0\rangle_h \quad (6)$$

where  $c_{\mathbf{k}+\mathbf{Q}\alpha} |0\rangle_h = 0$  and  $h_{\mathbf{k}\alpha} |0\rangle_h = 0$ . The particle-hole pair is in a triplet state with vanishing net spin projection along the quantization axis, has no charge and has

a wave vector equal to the antiferromagnetic wave vector. In the case of a FM, that wave vector would vanish.

Since we choose a different quantization axis in the two AF, we need to include a unitary transformation in spin space denoted by  $\mathbf{U}(\theta, \phi)$  to account for the fact that a  $\uparrow$  spin on one side of the junction is not the same as a  $\uparrow$  spin on the other side of the junction. Since the spin quantization axes are taken along the direction of the staggered magnetic moment  $\mathbf{S}$  on each side, the angles  $(\theta, \phi)$  correspond to the orientation of  $\mathbf{S}_R$  of the right AF expressed in the coordinate system of the left side of the junction; in Cartesian coordinate  $\mathbf{S}_L = |\mathbf{S}_L|(0, 0, 1)$  and  $\mathbf{S}_R = |\mathbf{S}_R|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ . The tunneling Hamiltonian then reads  $\hat{H}_T = (1/N) \sum_{\mathbf{k}\mathbf{q}\sigma\sigma'} (\mathbf{U}_{\sigma\sigma'} t_{\mathbf{k}\mathbf{q}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{q}\sigma'} + h.c.)$ . The spin flip terms come purely from the choice of different quantization axes on the left and on the right. There is no real spin flip in the tunneling process. Note also that the generators of the unitary transformation  $\mathbf{U}(\theta, \phi)$  are non-abelian gauge fields that are the analog of the electromagnetic gauge potentials in the ordinary Josephson effect. The former do not couple to the electromagnetic fields and they do not have a dynamics independent of that of the spins.

*Derivation of the equations of motion.* The staggered magnetic moment operator in the left lead is  $\hat{\mathbf{S}}_L = (\hbar/2) \sum_{\mathbf{k}\alpha\beta} c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger \sigma_{\alpha\beta} c_{\mathbf{k}\beta}$  where  $\sigma$  is a vector of Pauli matrices. In the broken symmetry state, the time evolution of  $\hat{\mathbf{S}}_L$  due to  $H_L$  is negligible. Since we also have  $[\hat{\mathbf{S}}_L, \hat{H}_R] = 0$  we find  $d\hat{\mathbf{S}}_L/dt = (1/i\hbar)[\hat{\mathbf{S}}_L, \hat{H}_T]$ , and thus  $d\hat{\mathbf{S}}_L/dt = -(i/2N) \sum_{\mathbf{k}\mathbf{q}} \sum_{\alpha\beta\delta} (\sigma_{\alpha\beta} \mathbf{U}_{\beta\delta} t_{\mathbf{k}\mathbf{q}} c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger c_{\mathbf{q}\delta} - h.c.)$  whose average  $\dot{\mathbf{S}}_L(t) \equiv \langle d\hat{\mathbf{S}}_L/dt \rangle$  is given by

$$\dot{\mathbf{S}}_L(t) = \frac{1}{N} \sum_{\mathbf{k}\mathbf{q}} \sum_{\alpha\beta\delta} \text{Im} \left[ \sigma_{\alpha\beta} \mathbf{U}_{\beta\delta} t_{\mathbf{k}\mathbf{q}} \langle c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger(t) c_{\mathbf{q}\delta}(t) \rangle \right], \quad (7)$$

where  $\langle \dots \rangle$  is the thermal statistical average with the full density matrix. Performing first order perturbation theory using  $\hat{H}_T$  as the perturbation, one obtains

$$\langle c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger c_{\mathbf{q}\delta} \rangle = -\frac{i}{\hbar} \int_{-\infty}^t dt' \langle [c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger(t) c_{\mathbf{q}\delta}(t), \hat{H}_T(t')] \rangle_0, \quad (8)$$

where the average  $\langle \dots \rangle_0$  is computed with  $H_0 = \hat{H}_L + \hat{H}_R$  (the unperturbed part of  $\hat{H}$ ). The operators on the right are in the interaction representation.

Once the commutator is evaluated in Eq. (8), one of the factorizations of the four-point correlation function involves products of correlation functions on the left and on the right leads such as  $\langle c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger(t) c_{\mathbf{k}\alpha}(t') \rangle_0 \langle c_{\mathbf{q}+\mathbf{Q}\delta}(t) d_{\mathbf{q}\delta}^\dagger(t') \rangle_0$ . Such correlation functions would vanish in a normal paramagnetic state. They are non-zero because of the broken symmetry. They represent interference in the tunneling process between momentum  $\mathbf{k} + \mathbf{Q}$  spin up particles and momentum  $-\mathbf{k}$  spin-down holes, in other words tunneling of charge zero spin one  $S^z = 0$  coherent particle-

hole pairs that have finite momentum and are present in the ground-state Eq.(5). In the case of the ordinary Josephson effect, one would find terms such as  $\langle c_{\mathbf{k}\sigma}^\dagger(t)c_{-\mathbf{k}-\sigma}^\dagger(t') \rangle_0 \langle d_{\mathbf{q}\sigma}(t)d_{-\mathbf{q}-\sigma}(t') \rangle_0$  that represent tunneling of coherent Cooper pairs.

In order to compute the averages  $\langle \dots \rangle_0$  in the broken symmetry states, we invert the Bogoliubov transformation Eq. (3). Assuming  $t_{\mathbf{k}\mathbf{q}} = t_{\mathbf{k}\mathbf{q}+\mathbf{Q}} = t_{\mathbf{k}+\mathbf{Q}\mathbf{q}+\mathbf{Q}}$ , we find

$$\sum_{\mathbf{k}\mathbf{q}} t_{\mathbf{k}\mathbf{q}} c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger d_{\mathbf{q}\delta} = \sum_{\mathbf{k}\mathbf{q}}^* \sum_{i,j \in \{c,v\}} t_{\mathbf{k}\mathbf{q}} (\Gamma_{\mathbf{k}\mathbf{q}}^{\alpha\delta})_{ij} \gamma_{\mathbf{k}\alpha}^{i\dagger} \gamma_{\mathbf{q}\delta}^j, \quad (9)$$

where we defined

$$\begin{aligned} (\Gamma_{\mathbf{k}\mathbf{q}}^{\alpha\delta})_{cc} &\equiv (u_{\mathbf{k}}u_{\mathbf{q}} + \sigma_{\alpha\alpha}^3 v_{\mathbf{k}}u_{\mathbf{q}} + \sigma_{\delta\delta}^3 u_{\mathbf{k}}v_{\mathbf{q}} + \sigma_{\alpha\alpha}^3 \sigma_{\delta\delta}^3 v_{\mathbf{k}}v_{\mathbf{q}}) \\ (\Gamma_{\mathbf{k}\mathbf{q}}^{\alpha\delta})_{cv} &\equiv (v_{\mathbf{k}}u_{\mathbf{q}} - \sigma_{\alpha\alpha}^3 u_{\mathbf{k}}u_{\mathbf{q}} + \sigma_{\delta\delta}^3 v_{\mathbf{k}}v_{\mathbf{q}} - \sigma_{\alpha\alpha}^3 \sigma_{\delta\delta}^3 u_{\mathbf{k}}v_{\mathbf{q}}) \\ (\Gamma_{\mathbf{k}\mathbf{q}}^{\alpha\delta})_{vc} &\equiv (u_{\mathbf{k}}v_{\mathbf{q}} + \sigma_{\alpha\alpha}^3 v_{\mathbf{k}}v_{\mathbf{q}} - \sigma_{\delta\delta}^3 u_{\mathbf{k}}u_{\mathbf{q}} - \sigma_{\alpha\alpha}^3 \sigma_{\delta\delta}^3 v_{\mathbf{k}}u_{\mathbf{q}}) \\ (\Gamma_{\mathbf{k}\mathbf{q}}^{\alpha\delta})_{vv} &\equiv (v_{\mathbf{k}}v_{\mathbf{q}} - \sigma_{\alpha\alpha}^3 u_{\mathbf{k}}v_{\mathbf{q}} - \sigma_{\delta\delta}^3 v_{\mathbf{k}}u_{\mathbf{q}} + \sigma_{\alpha\alpha}^3 \sigma_{\delta\delta}^3 u_{\mathbf{k}}u_{\mathbf{q}}). \end{aligned} \quad (10)$$

Similarly, let  $\tilde{H}_T$  denote the part of  $\hat{H}_T$  that does not commute with  $c_{\mathbf{k}+\mathbf{Q}\alpha}^\dagger d_{\mathbf{q}\delta}$ . It can be written as  $\tilde{H}_T = (1/N) \sum_{\mathbf{k}\mathbf{q}\sigma\delta'}^* \sum_{ij} \mathbf{U}_{\sigma\delta'}^* t_{\mathbf{k}\mathbf{q}} (\Gamma_{\mathbf{k}\mathbf{q}}^{\sigma\delta'})_{ij} \gamma_{\mathbf{q}\delta'}^{j\dagger} \gamma_{\mathbf{k}\sigma}^i$ . Substituting these expressions into Eq. (8), one finds

$$\begin{aligned} \dot{S}_L(t) = \frac{1}{N^2} \sum_{\mathbf{k}\mathbf{q}}^* \sum_{\alpha\beta\delta} \sum_{\sigma\delta'} \text{Im} \left[ -\frac{i}{\hbar} |t_{\mathbf{k}\mathbf{q}}|^2 \vec{\sigma}_{\alpha\beta} \mathbf{U}_{\beta\delta} \mathbf{U}_{\sigma\delta'}^* \int dt' e^{-0^+(t-t')} \times \right. \\ \left. \sum_{ij} (\Gamma_{\mathbf{k}\mathbf{q}}^{\alpha\delta})_{ij} (\Gamma_{\mathbf{k}\mathbf{q}}^{\sigma\delta'})_{ij} \left( \mathcal{G}_{\mathbf{k}\sigma\alpha}^{i<}(t'-t) \mathcal{G}_{\mathbf{q}\delta\delta'}^{j>}(t-t') - \mathcal{G}_{\mathbf{k}\sigma\alpha}^{i>}(t'-t) \mathcal{G}_{\mathbf{q}\delta\delta'}^{j<}(t-t') \right) \right] \end{aligned} \quad (11)$$

where  $\mathcal{G}_{\mathbf{k}(\mathbf{q})}^{i(>)} are the Keldysh Green functions in the left (right) lead. Their definitions are  $\mathcal{G}_{\mathbf{k}(\mathbf{q}),\alpha\beta}^{i<}(t,t') = i\langle \gamma_{\mathbf{k}(\mathbf{q})\beta}^{i\dagger}(t') \gamma_{\mathbf{k}(\mathbf{q})\alpha}^i(t) \rangle$  and  $\mathcal{G}_{\mathbf{k}(\mathbf{q}),\alpha\beta}^{i>}(t,t') = -i\langle \gamma_{\mathbf{k}(\mathbf{q})\alpha}^i(t) \gamma_{\mathbf{k}(\mathbf{q})\beta}^{i\dagger}(t') \rangle$ , respectively. Explicitly,$

$$\begin{aligned} \mathcal{G}_{\mathbf{k}\sigma\sigma'}^{i>}(t'-t) &= -i(1-f(E_k^i)) \exp[-iE_k^i(t'-t)/\hbar] \delta_{\sigma\sigma'}, \\ \mathcal{G}_{\mathbf{k}\sigma\sigma'}^{i<}(t'-t) &= if(E_k^i) \exp[-iE_k^i(t'-t)/\hbar] \delta_{\sigma\sigma'}, \end{aligned}$$

where  $f$  is the Fermi function. Eq. (11) for the staggered magnetic moment current through a tunnel junction is general. A bias could be included. We assume that there is no bias so there is no incoherent single-particle tunneling across the antiferromagnetic gap. Integrating over  $t-t'$  and performing the spin sum in (11), one finds

$$\dot{S}_L = I_c \hat{\mathbf{S}}_R \times \hat{\mathbf{S}}_L, \quad (12)$$

where  $\hat{\mathbf{S}}_{L(R)} = \mathbf{S}_{L(R)} / |\mathbf{S}_{L(R)}|$  and

$$I_c = \frac{8\Delta_L \Delta_R}{N^2} P \sum_{k\mathbf{q}}^* |t_{\mathbf{k}\mathbf{q}}|^2 \frac{f(E_{\mathbf{k}}) - f(-E_{\mathbf{k}})}{E_{\mathbf{k}}(E_{\mathbf{k}}^2 - E_{\mathbf{q}}^2)} \quad (13)$$

with  $P$  indicating principal part. A similar expression is found for the equilibrium spin current in the case of ferromagnetic tunnel junctions. Note that the sine function present in the standard Josephson case is replaced here by a cross product, which is a direct consequence of the vectorial nature of the order parameter.

For a symmetrical junction ( $\Delta_L = \Delta_R$ ), the same assumptions and procedure as Ref. [7] lead to the following

analytical result

$$I_c = \frac{\hbar}{e^2} R^{-1} \Delta(T) \tanh(\frac{1}{2}\beta\Delta(T)), \quad (14)$$

where  $R = \hbar / (4\pi e^2 D^2 |t|^2)$  is the (zero-temperature) normal-state resistance of the junction with  $D$  the density of state, which is assumed to be a constant. This expression for the temperature dependance of the critical current has the same form as that obtained by Ambegaokar and Baratoff [7] for a BCS superconductor, which is not surprising given the formal analogies [11].

By symmetry, the time derivative of the staggered magnetic moment on the right lead can be obtained by interchanging the  $L$  and  $R$  indices in Eq. (12). As a consequence, the staggered magnetic moments of the two AF precess about their (constant) sum  $\mathbf{S}_L + \mathbf{S}_R$  at a frequency  $\omega_0 = I_c |\mathbf{S}_L + \mathbf{S}_R| / |\mathbf{S}_L| |\mathbf{S}_R|$ . In the limit where  $\mathbf{S}_L$  and  $\mathbf{S}_R$  are nearly colinear, the effect should therefore lead to a uniform shift of order  $\omega_0$  in the antiferromagnetic resonance frequencies of each AF [12]. With  $|2\Delta/S| = U = 2\text{eV}$  and a normal state conductance of the order of the conductance quantum,  $R^{-1} = 2e^2/h$ , one finds a value of  $\omega_0$  of order  $10^{14}$  Hz, in other words in the visible. This frequency would be higher than the single particle gap and so would lead to much damping. Resistances that are many orders of magnitude larger are thus needed to bring its value down.

*AC spin-Josephson effect.* In the ordinary Josephson effect, the electromagnetic gauge potentials enter directly in the argument of the sine function. The present case is different. Each magnetic moment associated with a spin

couples to the magnetic field through the Zeeman term ( $H_Z = -g\mu_B \mathbf{B} \cdot \mathbf{S}$ ) where  $g$  is the gyromagnetic ratio and  $\mu_B$  the Bohr magneton (in insulators we can neglect terms coming from orbital motion).

Considering magnetic fields  $\mathbf{B}_L$  and  $\mathbf{B}_R$  applied respectively to the left- and right-hand sides of the junction, the Heisenberg equations of motion lead to the following equations of motion for the order parameters :

$$\begin{aligned}\dot{\mathbf{S}}_L &= -g\mu_B \mathbf{B}_L \times \mathbf{S}_L + I_c \hat{\mathbf{s}}_R \times \hat{\mathbf{s}}_L, \\ \dot{\mathbf{S}}_R &= -g\mu_B \mathbf{B}_R \times \mathbf{S}_R + I_c \hat{\mathbf{s}}_L \times \hat{\mathbf{s}}_R.\end{aligned}\quad (15)$$

The first term on the right side of the equality is merely the contribution of  $H_Z$  to the Heisenberg equation of motion. The second term is the tunneling contribution and has exactly the same form as the one we have already computed in the zero-field case, namely Eq.(12) and (13). To see this, return to Eq.(8). The only terms from  $H_T(t')$  that give a non-zero contribution to the average are of the form  $d_{\mathbf{q}\delta'}^\dagger(t')U_{\delta'\sigma'}^\dagger(t')c_{\mathbf{k}\sigma'}(t')$ . Since these operators are in the interaction representation and  $[H_0, H_Z] = 0$ , it is possible to write  $d_{\mathbf{q}\delta'}^\dagger(t') = d_{\mathbf{q}\delta'}^\dagger(\tau')\Lambda_{\delta\delta'}^{R\dagger}$  and  $c_{\mathbf{k}\sigma'}(t') = \Lambda_{\sigma'\sigma}^L c_{\mathbf{k}\sigma}(\tau')$  where  $\Lambda_{\sigma'\sigma}^{R(L)} = \exp[ig\mu_B \mathbf{B}_{R(L)} \cdot \boldsymbol{\sigma}(t' - t)/\hbar]_{\sigma'\sigma}$  and  $\tau'$  stands for the time evolution due to  $H_0$ . Similarly, one can show that the unitary transformation  $U_{\delta'\sigma'}^\dagger(t')$  can be written as  $\Lambda_{\delta'\delta}^R U_{\delta\sigma}^\dagger(t)\Lambda_{\sigma\sigma'}^{L\dagger}$ . We then obtain  $d_{\mathbf{q}\delta'}^\dagger(t')U_{\delta'\sigma'}^\dagger(t')c_{\mathbf{k}\sigma'}(t') = d_{\mathbf{q}\delta}^\dagger(\tau')U_{\delta\sigma}^\dagger(t)c_{\mathbf{k}\sigma}(\tau')$ , i.e all spin indices are at time  $t$ . Therefore, it is possible to factor the unitary transformation  $U_{\delta\sigma}^\dagger(t)$  out of the integral over  $t'$ , so that the rest of the calculation just follows the same path as in the zero-field case, thus leading to Eq. (15).

If  $\mathbf{B}_L = \mathbf{B}_R = \mathbf{B}$ , Eq.(15) implies that in the rotating frame  $\hat{\mathbf{u}} = \{\hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'\}$  defined by  $d\hat{\mathbf{u}}/dt = -g\mu_B \mathbf{B} \times \hat{\mathbf{u}}$ ,  $\mathbf{S}_L$  and  $\mathbf{S}_R$  still precess about their (constant) sum  $\boldsymbol{\Sigma} \equiv \mathbf{S}_L + \mathbf{S}_R$  at a frequency  $\omega_0$ . Returning to the static frame, Eq.(15) gives  $d\boldsymbol{\Sigma}/dt = -g\mu_B \mathbf{B} \times \boldsymbol{\Sigma}$  so that  $\mathbf{S}_L$  and  $\mathbf{S}_R$  undergo a motion of double precession.

In the ordinary Josephson effect, a constant electric potential difference  $V$  leads to a phase difference  $\varphi$  that depends linearly on time ( $\dot{\varphi} = 2eV/\hbar$ ). In the present case, an analog is found by computing the time dependence of the relative orientation  $\vartheta$  between  $\mathbf{S}_L$  and  $\mathbf{S}_R$ . Using Eq. (15) to compute  $d(\cos \vartheta)/dt = d(\hat{\mathbf{s}}_L \cdot \hat{\mathbf{s}}_R)/dt$ , we obtain

$$\dot{\vartheta}(t) = -g\mu_B \delta \mathbf{B} \cdot \hat{\mathbf{e}}(t), \quad (16)$$

where  $\delta \mathbf{B} \equiv \mathbf{B}_L - \mathbf{B}_R$  and  $\hat{\mathbf{e}}(t) \equiv \hat{\mathbf{s}}_L(t) \times \hat{\mathbf{s}}_R(t)$ . As in the ordinary Josephson effect, this expression does not contain explicitly the tunneling matrix element. Contrary to the ordinary Josephson effect, this equation is non-linear. Solving it numerically along with Eq. (15) in the case where the magnetic field vanishes on one side of the junction, one finds that the angle between  $\mathbf{S}_L$  and  $\mathbf{S}_R$  behaves as a sine-like function of time. The presence

of an additional constant magnetic field  $\mathbf{B}$  throughout the system adds a beat to this sine-like behavior. The gyromagnetic ratio  $g$  is material dependent. No such non-universal constant appears in the ordinary Josephson effect. The above discussion of the AC effect can be transposed for FM by replacing the staggered magnetic moment by the uniform one.

In summary, we found a Josephson-like equilibrium current of the staggered magnetic moment through a tunnel junction between two itinerant AF with non-collinear staggered magnetic moments. In analogy with the ferromagnetic case, this current is proportional to  $\mathbf{S}_R \times \mathbf{S}_L$  where  $\mathbf{S}_L$  and  $\mathbf{S}_R$  are the staggered magnetic moments on either sides of the junction. The effect comes from tunneling of coherent spin-one neutral particle-hole pairs with a net momentum and no net spin projection in the order parameter direction. There are important differences with the ordinary Josephson effect coming both from the different coupling of the electromagnetic fields and from the non-abelian nature of the broken symmetry. It would be interesting to perform magnetic resonance experiments to detect the effects we predicted.

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